Fourier Analysis

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Review.

(Isoperimetric inequality)

Thm 1. Let Γ be a C^1 simple closed curve in $\mathbb{R}^2$.

Set $A = \text{area}(\Omega)$, ℓ = length of Γ.

Then

$$A \le \ell^2/4\pi$$

and "=" holds if and only if Γ is a circle.

Pf. By a suitable scaling, we may assume $\ell = 2\pi$.

Parametrize Γ by its arclength, say,

$$\gamma(t) = (x(t), y(t)), \quad 0 \le t \le 2\pi,$$

$$x'(t)^2 + y'(t)^2 = 1$$

Using the Green Thm, we have

$$A = \oint_\Gamma x\,dy = \int_0^{2\pi} x(t)\, y'(t)\, dt$$

We need to show that $A \le \pi$ and "=" holds iff Γ is a circle.

It is equivalent to show that

$$\int_0^{2\pi} x(t)\, y'(t)\, dt \le \pi \quad \text{and "=" holds iff } \Gamma \text{ is a circle.}$$

For this purpose, we expand $x(t)$, $y(t)$ into their Fourier series on $[0, 2\pi]$.

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}, \quad y(t) = \sum_{n=-\infty}^{\infty} b_n e^{int}.$$

(the above Fouriers converge because $x(t)$, $y(t)$ are diff)

$$x'(t) \sim \sum_{n=-\infty}^{\infty} in\, a_n e^{int}, \quad y'(t) \sim \sum in\, b_n e^{int}.$$

($\widehat{f'}(n) = in \hat{f}(n)$)

By Parseval identity

$$\frac{1}{2\pi} \int_0^{2\pi} x'(t)^2\, dt = \sum_{n=-\infty}^{\infty} |in\, a_n|^2 = \sum_{n=-\infty}^{\infty} n^2 |a_n|^2.$$

$$\frac{1}{2\pi} \int_0^{2\pi} y'(t)^2\, dt = \sum_{n=-\infty}^{\infty} n^2 |b_n|^2.$$

Hence

$$1 = \frac{1}{2\pi}\int_0^{2\pi} x'(t)^2 + y'(t)^2 dt = \sum_{n=-\infty}^{\infty} n^2\left(|a_n|^2+|b_n|^2\right)$$

$(*)$

Also by the generalized Parseval identity,

$$\frac{1}{2\pi}\int_0^{2\pi} x(t)\, y'(t) dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, \overline{y'(t)}\, dt$$

$$= \langle x(t), y'(t)\rangle$$

$$= \sum_{n=-\infty}^{\infty} \hat{x}(n) \cdot \overline{\widehat{y'}(n)}$$

$$= \sum_{n=-\infty}^{\infty} a_n \, \overline{in\, b_n}$$

$$= \sum_{n=-\infty}^{\infty} -in\, a_n \overline{b_n}.$$

Hence $A = \int_0^{2\pi} x(t)\, y'(t)\, dt = 2\pi \cdot \sum_{n=-\infty}^{\infty} (-in\, a_n \overline{b_n})$.

So

$$A = 2\pi \left| \sum_{n=-\infty}^{\infty} (-in\, a_n \overline{b_n}) \right|$$

$$\leq 2\pi \sum_{n=-\infty}^{\infty} |n|\, |a_n|\, |b_n|$$

$$\leq 2\pi \sum_{n=-\infty}^{\infty} |n| \frac{|a_n|^2+|b_n|^2}{2}$$

$$\leq 2\pi \cdot \sum_{n=-\infty}^{\infty} |n|^2 \frac{|a_n|^2+|b_n|^2}{2}$$

$$= \pi \quad (\text{by } (*)).$$

This proves the isoperimetric inequality.

Next assume that $A = \pi$.

Clearly we have

① $|a_n| = |b_n|$ for all $n \neq 0$.

(since $|n|\,|a_n||b_n| = |n| \left(\frac{|a_n|^2 + |b_n|^2}{2} \right)$

② $|a_n| = |b_n| = 0$ for all $|n| > 1$.

(since $|n| \cdot \frac{|a_n|^2 + |b_n|^2}{2} = |n|^2 \cdot \frac{|a_n|^2 + |b_n|^2}{2}$)

Hence $x(t) = a_{-1} e^{-it} + a_0 + a_1 e^{it}$,

$y(t) = b_{-1} e^{-it} + b_0 + b_1 e^{it}$.

Since $x(t), y(t)$ are real,

$a_0 \in \mathbb{R}$, $a_{-1} = \overline{a_1}$, $b_{-1} = \overline{b_1}$, $b_0 \in \mathbb{R}$.

(check $a_0 = \frac{1}{2\pi} \int_0^{2\pi} x(t)\,dt \in \mathbb{R}$.

$a_{-1} = \frac{1}{2\pi} \int_0^{2\pi} x(t)\, e^{it}\, dt = \overline{\frac{1}{2\pi} \int_0^{2\pi} x(t) e^{-it}\, dt}$

$= \overline{a_1}$.)

Hence $|a_1| = |a_{-1}| = |b_1| = |b_{-1}|$

Recall $1 = \sum_{n=-\infty}^{\infty} n^2 \left(|a_n|^2 + |b_n|^2 \right)$. (by (*))

$$= |a_1|^2 + |b_1|^2 + |a_{-1}|^2 + |b_{-1}|^2$$

It follows that $|a_1| = |b_1| = |a_{-1}| = |b_{-1}| = \frac{1}{2}$

Now we see that

$$a_1 = \frac{1}{2} e^{i\alpha}, \quad b_1 = \frac{1}{2} e^{i\beta}$$

for some $\alpha, \beta \in [0, 2\pi)$.

Then

$$\begin{aligned} x(t) &= \overline{a_1} e^{-it} + a_0 + a_1 e^{it} \\ &= a_0 + \frac{1}{2} e^{-i(\alpha+t)} + \frac{1}{2} e^{i(\alpha+t)} \\ &= a_0 + \cos(\alpha+t). \end{aligned}$$

$$y(t) = b_0 + \cos(\beta+t).$$

Recall that

$$\begin{aligned} \pi = A &= 2\pi \sum_{n=-\infty}^{\infty} \left(-in\, a_n \overline{b_n} \right) \\ &= 2\pi \left(-i\, a_1 \overline{b_1} + i\, a_{-1} \overline{b_{-1}} \right) \\ &= 2\pi(-i) \left(\frac{1}{4} e^{i(\alpha-\beta)} - \frac{1}{4} e^{i(\beta-\alpha)} \right) \end{aligned}$$

$$= 2\pi(-i) \cdot \frac{1}{4} \cdot (2i) \sin(\alpha-\beta)$$

$$= \pi \sin(\alpha-\beta)$$

Hence $\sin(\alpha-\beta) = 1$. Since $\alpha, \beta \in [0, 2\pi)$, we have

$$\alpha - \beta = \frac{\pi}{2} \text{ or } -\frac{3\pi}{2}.$$

So

$$y(t) = b_0 + \cos(\beta + t)$$

$$= b_0 + \cos\left(\alpha + t - \frac{\pi}{2}\right)$$

$$\left(\text{or } \cos\left(\alpha + t + \frac{3\pi}{2}\right)\right)$$

$$= b_0 + \sin(\alpha + t)$$

Recall $x(t) = a_0 + \cos(\alpha + t)$.

Hence $\left(x(t) - a_0\right)^2 + \left(y(t) - b_0\right)^2 = 1$.

So Γ is a unit circle. $\square$

§ 4.3 Weyl's equidistribution Theorem.

Def. A sequence of numbers $(x_n)_{n=1}^{\infty} \subset [0,1)$ is said to be equidistributed in $[0,1)$ if for all $(a,b) \subset [0,1)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \cdot \# \left\{ 1 \leq n \leq N : \quad x_n \in (a,b) \right\} = b-a. \qquad (1)$$

Remark: the above limit is the proportion of (x_n) lying (a,b).

Example 1. Consider (x_n) given by

$$0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, \ldots$$

Take $(a,b) = (\frac{1}{3}, \frac{3}{8})$.

But $x_n \notin (a,b)$, so

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leq n \leq N : \quad x_n \in (a,b) \right\} = 0 \neq b-a.$$

So we conclude (x_n) is not equidistributed in $[0,1)$.

Thm 2. (Weyl) Let γ be an irrational number, $\gamma > 0$.

Then the sequence

$$\left(\{n\gamma\}\right)_{n=1}^{\infty}$$

is equidistributed in $[0,1)$.

Here $\{n\gamma\}$ denotes the fractional part of $n\gamma$.

(e.g if $x = 2.345\cdots$, then $\{x\} = .345\cdots$)

Remark: Kronecker proved that

$$\left(\{n\gamma\}\right)_{n=1}^{\infty}$$

is dense in $[0,1)$ if γ is irrational.

- For $(a,b) \subset [0,1)$, let us define $\chi_{(a,b)} : [0,1) \to \mathbb{R}$ by

$$\chi_{(a,b)}(x) = \begin{cases} 1 & \text{if } x \in (a,b) \\ 0 & \text{otherwise.} \end{cases}$$

$\chi_{(a,b)}$ is called the characteristic function of (a,b).

Then it is direct to check

$$\#\{ 1 \leq n \leq N : x_n \in (a,b) \}$$

$$= \sum_{n=1}^{N} \chi_{(a,b)}(x_n)$$

In this way, we see that ① is equivalent to

② $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(x_n) = b-a.$$

for all $(a,b) \subset [0,1)$

To prove the theorem of Weyl, it is enough to show that for any irrational number γ,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(\{n\gamma\}) = b-a, \quad \forall (a,b) \subset [0,1).$$

We call extend $\chi_{(a,b)}$ to be a 1-periodic function on $\mathbb{R}$

Then, we can write

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) = b-a, \quad \forall (a,b) \subset [0,1).$$